Quantum Mechanical Foundations of Causal Entropic Forces

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Abstract. The theory of Causal Entropic Forces was introduced to explain the emergence of intelligence as a phenomenon in physical systems. Although the theory provides illustrations of how behavior shaped by causal entropic forces resembles human cognitive niche in specific simple settings, the theory leaves open some important questions. First, the definition of causal entropic forces, in terms of actions that maximize the statistical diversity of future paths a system can take, makes no connection with concepts of knowledge and rationality traditionally associated with intelligence. Second, the theory does not explain the origins of such path based forces in classical thermodynamic ensembles. This paper addresses both these issues using the principles of open system quantum mechanics, quantum statistics and the Hamiltonian theory of Dynamic Economics. The construction finally arrived at is much more general than the notion of entropic forces and shows how maximizing future path diversity is closely related to maximizing a particular utility function over a sequence of interactions till the system attains thermodynamic equilibrium.

Keywords: Entropic Forces, Statistical Diversity, Open Quantum Systems, Utility Maximization

One of the fundamental problems in Artificial Intelligence existing right from the inception of the field is the lack of a precise formulation. The most widely studied AI architectures involve logical agents, goal based agents and utility maximizing agents. The latter approach, also termed as the economic approach to rationality, proposes that intelligence can be referred to as the agent's ability to maximize a utility or value function over the sequence of states it will see in its lifetime, given transition probabilities for combinations of states and actions. In this approach the agent receives a reward for each state it sees, based on a reward function over the sample space of states, and its job is to maximize the predicted future sum of these rewards or the utility of its action. The problem here lies in getting a consensus on the global preference order on the utilities of actions in a particular state, which has been discussed in great depth in existing literature [2].

The paper on Causal Entropic Forces [1] proposes a first step towards establishing a connection between thermodynamic entropy maximization over a future path and intelligent behavior. It purports the idea that general causal entropic forces can result in spontaneous emergence of intelligent behavior in simple physical systems without the need for explicitly specifying goals or utilities. However, there are two potential problems with this paper – a) It hypothesizes a force, dependent on the statistical diversity of future paths up to a finite time horizon, as against those based on instantaneous entropy production which are widely studied in statistical thermodynamics. The conditions under which such forces might exist have not been established. b) Although it does illustrate, with examples, emergence of 'human-like' behavior in the narrow sense of tool use and walking abilities, it is far from establishing a clear relation between the traditional notion of intelligence and existence of forces that maximize path diversity. This paper attempts to resolve both the above mentioned issues in the context of open quantum systems and the Hamiltonian theory of dynamic economics [6]. The next section provides a very brief introduction to Quantum Mechanics for the non physicists.

1 Background

1.1 Quantum Mechanics

In non-relativistic QM, all matter in the universe is expressed in the form of a wavefunction which associates nonzero probability amplitude to every coordinate in the combined configuration space of all particles (as against the conventional Euclidean space, in the configuration space every point corresponds to the degrees of freedom of all particles). The time evolution of this universal wavefunction ψ is determined by the Schrodinger wave equation:

$$\imath\hbar\frac{\partial\psi}{\partial t} = H\psi \tag{1}$$

where H is the quantum Hamiltonian. It is a linear operator with eigenvalues as possible values of energy of the system.

1.2 Density Matrix Formalism and the Canonical Ensemble

If the values of all commuting observables are not known, we have more than one wave-function describing the system. Under this condition, the system state is represented by a density matrix which describes a statistical ensemble of wave-functions:

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k| \tag{2}$$

The density operator is positive definite so it can always be diagonalized in an eigenbasis. The time evolution of the density matrix is then given by the von-Liouville equation which is the quantum counterpart of the classical Liouville equation in statistical mechanics:

$$\imath\hbar\frac{\partial\rho}{\partial t} = [H,\rho] \tag{3}$$

If the universe is separated into a system and an environment which are in thermal equilibrium (only able to exchange energy), the density matrix is given by the celebrated Gibbs state:

$$\frac{e^{-\beta H}}{\mathrm{Tr}\left(e^{-\beta H}\right)} \tag{4}$$

Here β is the inverse temperature of the reservoir and H is the system Hamiltonian.

2 Causal Entropic Forces

The paper [1] proposes a first step towards establishing a connection between path entropy maximization and intelligent behavior. It purports the idea that general causal entropic forces can result in spontaneous emergence of intelligent behavior in simple physical systems without the need for explicitly specifying goals or utilities. It suggests a potentially general thermodynamic model of adaptive behavior as a non-equilibrium process in open systems. A nonequilibrium physical system's bias towards maximum instantaneous entropy production is reflected by its evolution toward higher-entropy macroscopic states, a process characterized by the formalism of entropic forces. The instantaneous entropic force on a canonical ensemble associated with a macrostate partition is given by:

$$F = T\boldsymbol{\nabla}_{\boldsymbol{X}} S(X) \tag{5}$$

where T is the reservoir temperature and $S(\mathbf{X})$ is the entropy associated with a macrostate $\mathbf{X}$ and $\mathbf{X_0}$ is the present macrostate. In order to uniformly maximize entropy production between present and a future time horizon, the author proposes generalized causal entropic forces over paths through the configuration space rather than over the instantaneous configuration (or the ensemble). He then defines the causal path entropy of a macrostate $\mathbf{X}$ with the current system state x(0) as:

$$S_C(X,\tau) = -k_B \int \Pr\big(x(t)\big|x(0)\big) \ln\, \Pr\big(x(t)\big|x(0)\big)\, dx(t) \tag{6}$$

where Pr(x(t)|x(0)) denotes the conditional probability of the system evolving through the path x(t) assuming the initial system state x(0), integrating over all possible paths taken by the open systems' environment during the same interval. A path-based causal entropic force F corresponding to (6), can be expressed as:

$$F(X_0,\tau) = T_C \boldsymbol{\nabla}_{\boldsymbol{X}} S_C(X,\tau)|\mathrm{X}_0 \tag{7}$$

where T_C is a causal path temperature that parametrizes the system's bias toward macrostates that maximize causal entropy. The remaining part of the causal path force derivation takes in to account specific assumptions about the environment being a heat bath at temperature T_r and the environment being coupled to only a few forced degrees of freedom. The environment periodically rethermalizes these forced degrees of freedom with a period ϵ. Further, the temporal dynamics of the system are taken to be Markovian in nature giving the path probability as:

$$\Pr\big(x(t)\big|x(0)\big) = \left(\prod_{n=1}^{N-1} \Pr\big(x(t_{n+1})\big|x(t_n)\big)\right) \Pr\big(x(\epsilon)\big|x(0)\big) \tag{8}$$

Making use of all these assumptions, the author finally arrives at the following path dependent force (Specific details not mentioned here. The reader is referred to [1] for the detailed derivation):

$$F_j(X_0, \tau) = -\frac{2T_C}{T_R} \int f_j(0) \cdot \Pr\,(x(t)|x(0)) \ln \Pr\,(x(t)|x(0)) dx(t) \tag{9}$$

The effect of the above force can be seen as driving the forced degrees of freedom with a temperature dependent strength in an average of short term directions $f_j(0)$, weighted by the diversity of long term paths $[-\Pr\,(x(t)|x(0)) \ln \Pr\,(x(t)|x(0))]$, that they make reachable, where the path diversity is measured over all degrees of freedom of the system.

3 Causal Entropic forces in a Quantum Universe

3.1 Projective or Von-Neumann Interactions

If the time horizon for evaluating the causal entropic force $\tau \to \infty$, equation (6) gets modified to the following, owing to the celebrated asymptotic equi-partition property:

$$F_j(X_0, \tau) \propto \frac{2T_C \cdot [n\bar{H}(X)]}{T_R} \int f_j(0) \cdot \Pr(x(\epsilon)|x(0)) dx(\epsilon)\,, \quad n = \left\lfloor \frac{\tau}{\epsilon} \right\rfloor \tag{10}$$

where $\bar{H}(X)$ is the entropy rate of the Markov process. This is a valid assumption to make if the Markov chain is ergodic and stationary. Having established this let us move onto the development of the Quantum analogue.

As explained in [5], measurements and actions can be treated as system bath interactions in two mutually non-commuting eigenbases. Under the Born-Markov approximation, the temporal evolution of the reduced system (with bath degrees traced away) is described by the Lindblad Master equation:

$$\frac{\partial \rho_S}{\partial t} = -\frac{\iota}{\hbar}[H_S + H_{LS}, \rho_S] - \frac{1}{2}\sum_k \mu_k (A_k^\dagger A_k \rho_S + \rho_S A_k^\dagger A_k - 2A_k \rho_S A_k^\dagger) \tag{11}$$

As in the case of Langevin dynamics, the bath thermalizes coupled degrees of freedom of the system in the pointer basis (the Lindbladian basis A_k for projective interactions) after each interaction. We also assume the density matrix to be initially diagonal in the measurement eigenbasis $|x_i\rangle$. The expectation value of the force on the ensemble is given by:

$$\langle F \rangle = \mathrm{Tr}\left(\frac{\partial \rho}{\partial t} \cdot P\right) \tag{12}$$

Here P is the momentum operator. Now, this ensemble force can be simplified as under:

$$\langle F \rangle = \sum_i \rho_i \cdot \langle F_i \rangle \tag{13}$$

Furthermore, we will assume that the action Lindbladian is PVM (projective valued measure) of the action observable. From equations (12) and (13), using explicit form of the Lindblad generator and common dissipation rates μ for all components forming the ensemble, the non-unitary component of the force (neglecting Hamiltonian dynamics) in a measurement eigenstate $|x_i\rangle$ is given by:

$$\langle F_i \rangle = \frac{1}{2} \mu \sum_k (|\langle x_i | y_k \rangle|^2 \langle y_k | P | y_k \rangle - \langle y_k | x_i \rangle \langle x_i | P | y_k \rangle) \tag{14}$$

Here, $|y_k\rangle$ is an action basis (substituted for the Lindblad operator eigenbasis) and μ is Fourier transform of the time correlation function of corresponding Bath operators [4]. With some algebra, we obtain:

$$\langle F_i \rangle = \frac{1}{2} \mu \sum_k |\langle x_i | y_k \rangle|^2 \left[\langle y_k | P | y_k \rangle - \langle x_i | P | x_i \rangle \right] \tag{15}$$

Notice the resemblance of the above equation with (10). The contribution of each action eigenstate to this force is proportional to statistical diversity (parametrized by the Born probability $|\langle y_k | x_i \rangle|^2$) of all future paths resulting from the action. The above equation explains the origins of such path based causal forces for projective Markovian interactions as the time horizon for evaluation of posterior path probabilities $\tau \rightarrow \infty$ in (9).

3.2 Generalized Interactions

In case of projective interactions, the Von Neumann entropy $-k_B \mathrm{Tr}(\rho \ln \rho)$ of the system always increases [5]. So the system does not approach a goal state as against that illustrated in [1]. We turn to generalized action interactions (positive operator valued measures that are generalizations of projective valued measures) in this section which can decrease an open system's entropy at the cost of entropy of the bath, enabling the system to approach a goal state asymptotically as the system attains equilibrium. We will also assume that the average internal energy of the system is initially higher than reservoir temperature. For non equilibrium processes, the total entropy production rate is given by the detailed balance relation (second law for open systems):

$$\sigma = \frac{dS}{dt} + J, \quad \sigma \geq 0 \tag{16}$$

where σ is the total entropy production rate and J is the entropy flux owing to heat exchange between the system and its environment.

3.2.1 Maximizing Path Diversity

In the current context, the diversity of future paths can be termed as the maximum entropy that can be produced from the present to a future time horizon. Given the density operator ρ at the present time, the future path diversity is given by:

$$D = \Gamma + k_B \mathrm{Tr}(\rho \ln \rho) \tag{17}$$

where Γ is the maximum attainable future system entropy. Integrating the entropic balance relation (16) over the time span to reach equilibrium, one gets:

$$\Delta\sigma = \Delta S + \Delta J = \Delta S - \frac{\Delta H}{T} \tag{18}$$

Let the entropy at time $t = 0$ be S_0 and the current absolute entropy be S_t. Then making use of the balance relation in conjunction with the second law, we get:

$$D \leq (\Gamma - S_0) - \frac{\Delta H}{T} \tag{19}$$

Because the average internal energy of the system is assumed to be initially higher than the bath temperature, as the system progresses towards equilibrium heat is dissipated to the bath. So the heat exchange term in the above inequality is negative. Secondly Γ is the maximum attainable future system entropy, so the first term on the right side of the inequality is positive making the quantity on the right side strictly positive and it is the maximum value that can be attained for the future path diversity. As can be inferred, this value is approached when the total entropy production over time ($\Delta\sigma$) is extremely small.

4 Causal Entropic Forces and Maximizing Expected Utility

4.1 Hamiltonian Theory of Dynamic Economics

Having laid down the foundations of path based forces in Quantum Mechanics for open systems, we now develop the relation between economic utility and path diversity. For the problem of consumption-optimal growth with positive rate of time discount $\alpha > 0$, the equations of motion are [6]:

$$k \in \partial_Q H(Q, k) \tag{20}$$

$$\dot{Q} \in -\partial_k H(Q, k) + \alpha Q \tag{21}$$

where, k is the initial endowment vector and Q is the vector of capital goods prices and H is the system Hamiltonian representing the production technology. The optimal steady state at the equilibrium (Q^*,k^*) is given by:

$$0 \in \partial_Q H(Q^*, k^*) \quad (22)$$

$$0 \in -\partial_k H(Q^*, k^*) + \alpha Q^* \quad (23)$$

From [7], the optimal path to the steady state for a diffusion process is given by the one that maximizes the following discounted expected utility over (possibly) infinite sequence of interventions:

$$\max E\left[\sum_{i=1}^{\infty} U(c_i) e^{-\int_0^{\tau_i} \alpha(S_t)dt}\right] \quad (24)$$

where c_i is the consumption at time step i, S_t is the current value of quantity being consumed and U is the agent's utility from consumption c_i. We can assume the rate of time discount α to be constant for the sake of simplicity. According to [7], if the time discount rate α is given to be quite large, the optimal policy is Markovian in nature and is characterized by a control region, a complementary continuation region and a set of optimal actions that can be taken in control region. For the discrete time Markovian policy, equation (24) reduces to the well known Bellman equation:

$$V(c_i)^* = \max E\left[U(c_i) + V(c_{i+1})^* e^{-\int_{\tau_i}^{\tau_{i+1}} \alpha(S_t)dt}\right] \quad (25)$$

which can be rewritten as:

$$V(c_i)^* = \max \sum_{c_i}\left[\Pr(c_i)U(c_i) + \Pr(c_i)V(c_{i+1})^* e^{-\int_{\tau_i}^{\tau_{i+1}} \alpha(S_t)dt}\right] \quad (26)$$

4.2 Economic Utility and Entropy

Using standard form of the Lindblad Dissipator and assuming characteristic dissipation rates μ to be equal for all components, the rate of change of expectation value of $(-\ln\rho)$ in the system energy eigenbasis $|i\rangle$ is given by:

$$-\frac{\partial \langle i|\rho \ln\rho)|i\rangle}{\partial t} = -\mu\left\langle i\middle|\sum_{\omega}\sum_{\alpha,\beta}\left[A_\beta(\omega)\rho A_\alpha^\dagger(\omega)\ln\rho - \frac{1}{2}\{\rho, A_\alpha^\dagger(\omega)A_\beta(\omega)\}\ln\rho\right]\middle|i\right\rangle \quad (27)$$

where $\{*,*\}$ represents the anti-commutator. It is known from the theory of Markovian master equation for open systems that:

$$\left[H_s, A_\alpha^\dagger(\omega)A_\beta(\omega)\right] = 0 \quad (28)$$

So, equation (27) can be written as:

$$-\frac{\partial \langle i|\rho\ln\rho|i\rangle}{\partial t} = -\mu \cdot f(t) + \mu K \cdot \langle i|\rho\ln\rho|i\rangle \quad (29)$$

where,

$$K = \sum_{\alpha,\beta} \langle i | A_\alpha^\dagger A_\beta | i \rangle \quad (30.a)$$

$$f(t) = \left\langle i \middle| \sum_\omega \sum_{\alpha,\beta} \left[A_\beta(\omega)\rho A_\alpha^\dagger(\omega)\ln\rho - \frac{1}{2}\left[\rho A_\alpha^\dagger(\omega) A_\beta(\omega), \ln\rho\right]\right] \middle| i \right\rangle \quad (30.b)$$

Equation (29) is a first order linear differential equation with the general solution:

$$-\langle i|\rho\ln(\rho)|i\rangle\,(t) = [-\mu \int_0^t e^{\mu K t} f(t)dt + S_0]e^{-\mu K t} \quad (31)$$

So, the change in Von Neumann Entropy in an action interaction is given by:

$$-\Delta S = k_B \sum_i \left[\gamma + e^{-\mu K \tau_d} \cdot \mu \int_0^{\tau_d} e^{\mu K t} f(t)dt\right] \quad (32)$$

where $\gamma = S_0(1 - e^{-\mu K \tau_d})$ and τ_d is the characteristic decoherence time. Negative of change in entropy is used in (32) owing to the assumption that the system entropy decreases in an action interaction as was mentioned earlier. Notice the resemblance of (32) with the Bellman equation (26). From (18), it can be inferred that this situation is achieved when the total entropy production for an action interaction is extremely small. This is precisely the condition that we arrived at while trying to maximize future path diversity D (17) in the previous section, which suggests that both the problems of maximizing path diversity and maximizing expected utility are duals of each other in the proposed scheme. In terms of thermodynamic utility, this condition translates to the minimum reduction in the amount of Free energy of a system as it attains equilibrium. Free energy of a system corresponds to the maximum useful work that can be extracted from the system. As the system attains equilibrium with the heat bath, the system entropy attains its minimum value leading to emergence of an optimal stationary goal state.

5 Conclusions and Future Scope

Theory of causal entropic forces is based on the idea of maximizing causal path entropy by evaluating path probabilities upto a finite time horizon τ instead of greedily maximizing instantaneous entropy production [1]. However, adhering to classical thermodynamics, one cannot fully explain the origins of such a path based force acting on a macrostate of a statistical ensemble. Using the interaction model proposed in [5], we treat measurements and actions as system environment interactions in two non commuting eigenbases. We establish how such a force might originate in the Quantum mechanical framework for projective interactions. However, projective interactions result in increase in Von Neumann entropy of the system, in which case the system never approaches a singular goal state. So we turn to generalized interactions in order to explain maximization of path diversity when the system entropy decreases on each action interaction at the cost of entropy of the bath using detailed balance relation. We also show how under a suitable choice of utility function, maximizing path diversity and maximizing expected utility are duals of each other.

Future work would include determining the common criterion for optimality for maximizing path diversity and expected utility in the presented scheme in terms of system-environment coupling strength and properties of action operators that lead to non-projective Lindbladians, which in turn allow decrease in system entropy. In [8], the authors describe application of displaced oscillator variational ansatz to Caldeira Leggett model for Brownian particle in a box coupled to an ohmic dissipative environment. They show, with the help of numerical renormalization group techniques, how for a critical system-environment coupling strength, the particle gets localized to the center of the box which is analogous to the illustration of Brownian particle behavior under causal entropic forces [1]. A generalization of analysis in [8] would be necessary to arrive at the precise conditions necessary for emergence of optimal goal states in dissipative systems as predicted by theory of causal entropic forces. Once these conditions are established, it would be possible to explain intelligence as emergent phenomenon in general non-equilibrium thermodynamic systems and processes.

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