Quantum Mechanical Foundations of Causal Entropic Forces

Swapnil Shah
Department of Electrical and Computer Engineering
North Carolina State University, USA

NC STATE UNIVERSITY

“As the number of experts increase, each specialty becomes all the more self-sustaining and self-contained. Such balkanization carries scientific thought farther away from natural philosophy which, intellectually, is the meaning and goal of science”

- Isidor Isaac Rabi

Motivation and Goals

- Establish concrete mathematical relation between maximizing path diversity and notions of rationality traditionally associated with ‘intelligence’
- Understand the microscopic origins of forces that maximize diversity of future system paths
- Understand role of critical dynamics and long range correlations in human brain organization and behavior
- Understand the mechanisms behind self-organization of complex structures capable of sophisticated behavior such as cognition in systems governed solely by the laws of physics
- Use the knowledge of these mechanisms to enhance the capabilities of the contemporary cognitive architectures

Outline

- Causal Entropic Forces
- Open Quantum Systems
  - Markovian Master Equation
  - Projective vs. General Interactions
- Path Diversity and Expected Utility
  - Entropy Balance Relation and Maximum Path Diversity
  - Hamiltonian Theory of Dynamic Economics
  - Economic Utility and Entropy
- Current Work
  - Analysis of Brownian Particle in a Box – Displaced Oscillator Ansatz
  - Emergence of SOC (Self Organized Criticality) in proposed framework
- Conclusions

Causal Entropic Forces[1]

- Based on the idea of maximizing entropy production over finite duration paths

- Entropy over paths through configuration space:

$$S_C(X,\tau) = -k_B \int \Pr(x(t) \mid x(0)) \ln \Pr(x(t) \mid x(0))\, dx(t)$$

- Path based Entropic Force:

$$F(X_0,\tau) = T_C \nabla_X S_C(X,\tau) \mid X_0$$

- Environment as a heat bath at temperature T_r coupled only to a few degrees of freedom of the system

- Formulation of Path based Entropic Force under Markovian Langevin dynamics:

$$F_j(X_0,\tau) = -\frac{2T_C}{T_R} \int f_j(0) \cdot \Pr(x(t) \mid x(0)) \ln \Pr(x(t) \mid x(0))\, dx(t)$$

- Asymptotic Equipartition Property for horizon $\tau \to \infty$

$$\boxed{F_j(X_0,\tau) \propto \frac{T_C \cdot n\overline{H}(X)}{T_R} \int f_j(0) \cdot \Pr(x(\varepsilon) \mid x(0))\, dx(t), \quad n = \left\lfloor \frac{\tau}{\varepsilon} \right\rfloor}$$

[1]*Courtesy: 'Causal Entropic Forces', A.D. Gross & C.E. Freer*

Open Quantum Systems – Projective Interactions

- Density Matrix Evolution – Von Neumann-Liouville Equation

$$\imath\hbar \frac{\partial \rho}{\partial t} = [H, \rho]$$

- Lindblad Master Equation – Quantum Markov Master Equation:

$$\frac{\partial \rho_S}{\partial t} = -\frac{\imath}{\hbar}[H_S + H_{LS}, \rho_S] - \frac{1}{2}\sum_k \mu_k \left(A_k^{\uparrow} A_k \rho_S + \rho_S A_k^{\uparrow} A_k - 2A_k \rho_S A_k^{\uparrow}\right)$$

- Expectation Value of Force:

$$\langle F \rangle = \mathrm{Tr}\left(\frac{\partial \rho}{\partial t} \cdot P\right)$$

$$\langle F \rangle = \sum_i \rho_i \cdot \langle F_i \rangle$$

- Common Dissipation rate μ and Projective Valued Measure (PVM) Lindbladians:

$$\boxed{\langle F_i \rangle = \frac{1}{2}\mu \sum_k \left|\langle x_i | y_k \rangle\right|^2 \left[\langle y_k | P | y_k \rangle - \langle x_i | P | x_i \rangle\right]}$$

- Resemblance with the path based Entropic Force

Open Quantum Systems – General Interactions

- Von-Neumann Entropy always increases for Projective Lindbladians

- POVM (Positive Operator Valued Measure) Lindblad operators for general interactions:

$$\sum_i F_i = \mathrm{I}_H \qquad\qquad F_i = A_i^{\dagger} A_i$$

- System Entropy can decrease at the cost of Entropy of the bath

- Entropy balance relation (Second law for open systems)

$$\sigma = \frac{dS}{dt} + J, \qquad \sigma \geq 0$$

- Assumptions:

➢ System evolves as a non-equilibrium dissipative process

➢ The POVM Lindbladian for the system-bath interaction decreases the Von-Neumann Entropy of the system to allow a stationary goal state to be approached

➢ Average kinetic energy per molecule of the System is initially higher than reservoir temperature

Maximizing Path Diversity

- Future path diversity as maximum entropy that can be produced from the present state to a future time horizon:

$$D = \Gamma + k_B \mathrm{Tr}(\rho \ln \rho)$$

- From the Entropy Balance relation:

$$\Delta\sigma = \Delta S - \frac{\Delta H}{T}$$

- Maximum path diversity:

$$\boxed{D \le (\Gamma - S_0) - \frac{\Delta H}{T}}$$

- The system is dissipating heat to the reservoir and *Γ* is the maximum attainable future system entropy

- It can easily be inferred that this maximum value of diversity is approached when the entropy production from the initial to the present state (Δσ) is minimum.

Hamiltonian Theory of Dynamic Economics[2,3]

- For the problem of consumption-optimal growth with positive rate of time discount $\alpha > 0$, the equations of motion are:

$$k = \partial_Q H(Q,k)$$
$$\dot{Q} = -\partial_k H(Q,k) + \alpha Q$$

- H is the system Hamiltonian representing the production technology

- Optimal path to the steady state is given by the one that maximizes following discounted expected utility over (possibly) infinite sequence of interventions:

$$\max E\left[\sum_{i=1}^{\infty} U(c_i)\, e^{-\int_0^{\tau_i} \alpha(S_t)\,dt}\right]$$

- For large α, the optimal policy is Markovian in nature and is characterized by a control region, a complementary continuation region and a set of optimal actions that can be taken in control region

- For the discrete time Markovian policy, it reduces to the well known Bellman equation:

$$V(c_i)^* = \max \sum_{c_i} \left[\Pr(c_i) U(c_i) + \Pr(c_i) V(c_{i+1})^* e^{\int_{\tau_i}^{\tau_{i+1}} \alpha(S_t)\,dt} \right]$$

[2]*Courtesy: 'Hamiltonian Approach to Dynamic Economics', D. Cass and K. Shell*

[3]*Courtesy: 'Optimal Consumption of a generalized Geometric Brownian Motion with Fixed and Variable Intervention Costs', S. Baccarin*

Economic Utility and Entropy

- Repeated Interaction model – Actions in the control region, Measurements in the continuation region (Shah 2013)

- Lindblad Dynamics yield the following in the energy eigenbasis:

$$-\frac{\partial\langle i|\rho\ln\rho|i\rangle}{\partial t} = -\mu\cdot f(t) + \mu K\cdot\langle i|\rho\ln\rho|i\rangle$$

$$f(t) = \langle i|\sum_{\omega}\sum_{\alpha,\beta}\left[A_\beta(\omega)\rho A_\alpha^\uparrow(\omega)\ln\rho - \frac{1}{2}\left[\rho A_\alpha^\uparrow(\omega)A_\beta(\omega),\ln\rho\right]\right]|i\rangle \qquad K = \sum_{\alpha,\beta}\langle i|A_\alpha^\uparrow A_\beta|i\rangle$$

- The first order linear DE has the solution:

$$-\langle i|\rho\ln\rho|i\rangle(t) = \left[-\mu\int_0^{\tau_d} e^{\mu Kt} f(t)dt + S_0\right]e^{-\mu Kt}$$

- Change in Von Neumann Entropy in an action interaction takes form of Bellman Equation:

$$\boxed{-\Delta S = k_B\sum_i\left[\gamma + e^{-\mu K\tau_d}\cdot\mu\int_0^{\tau_d} e^{\mu Kt} f(t)dt\right] \qquad \gamma = S_0\left(1-e^{\mu K\tau_d}\right)}$$

- Under the assumptions made, the maximum of the ‘Entropic Utility’ above is obtained when entropy production per interaction (Δσ) is minimum.

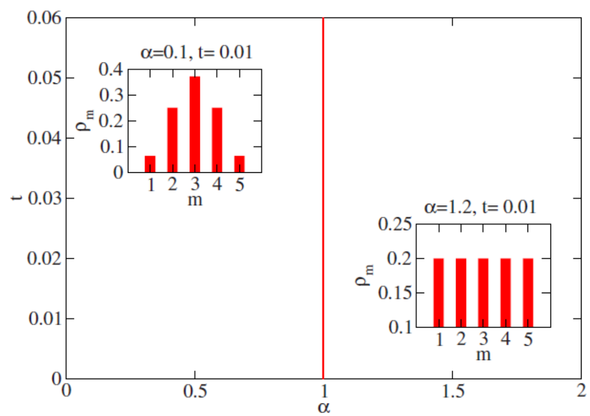
- Thus maximizing path diversity corresponds directly to maximizing the Entropic Utility

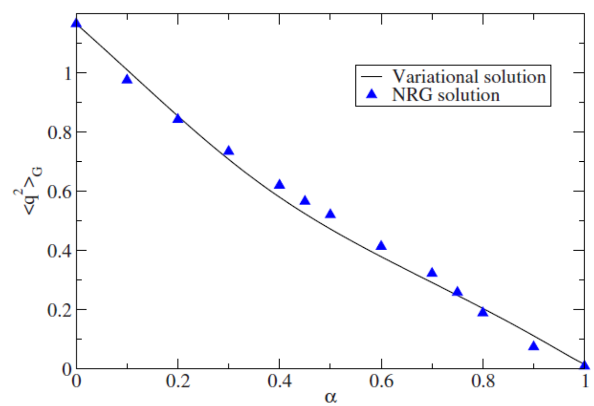
Current Work – Quantum Brownian Particle in a Box[4]

- Recent studies on phase diagram of particle confined to a finite binding chain coupled to ohmic dissipative bath

- Caldeira – Leggett model for general dissipative dynamics of a quantum particle interacting with a heat bath

- Hard wall boundary conditions affect the phase diagram of the confined particle





- The phase localization is sharpest when the system-bath coupling is critical (the phase transition point)

- Results agree with behavior of confined brownian particle under causal entropic forces

[4]*Courtesy: 'Phase Diagram of the Dissipative Quantum Particle in a Box', J. Sabio, L.Borda, F. Guinea and F. Sols*

Current Work – Emergence of Criticality

- Various studies in neuro-imaging suggest link between effective brain dynamics and critical behavior of physical systems (Kitzbichler et al. 2009)

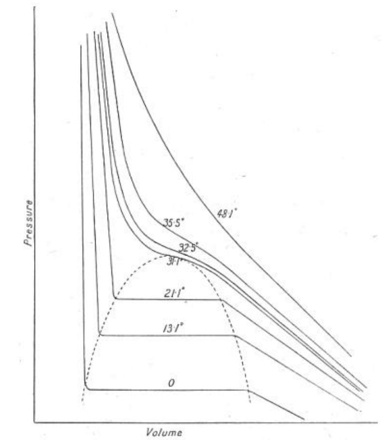
- System – Bath coupling μ_C determines system evolution when all other parameters held constant, similar to the ratio (T_C / T_R) in causal entropic forces

- Optimizing Entropic Utility in conjunction with First law yields:

$$\left(\frac{dU}{dS}\right)_{\mu_C} = T, \quad dU = TdS - PdV \quad \therefore \left(\frac{dV}{dS}\right)_{\mu_C} = 0$$

- Condition of Thermodynamic stability:

$$dF = -SdT - PdV = 0$$



- Using Total differential of Entropy at constant Pressure P and number of particles N with the above conditions yields:

$$\boxed{\Lambda_V = T\left(\frac{\partial S}{\partial V}\right)_{T,\mu_C} \to 0 \qquad C_V = T\left(\frac{\partial S}{\partial T}\right)_{V,\mu_C} \to \infty \qquad K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,\mu_C} \to \infty \qquad \left(\frac{\partial P}{\partial V}\right)_{T,\mu_C} \to 0}$$

- The stable point is indeed the critical point of phase transition as the latent heat of transition $(\Lambda_V) \to 0$ at the point.

- A more concrete and rigorous analysis using critical exponents furnished by numerical renormalization group techniques is under way

Conclusions

- Origin of path based entropic forces explained for Markovian projective interactions in nonequilibrium dissipative processes
- Established duality between maximizing expected utility (Entropic Utility) and maximizing path diversity
- Importance of system-bath coupling strength in determining system behavior and evidence of critical system dynamics at the optimal coupling.
- No need to explicitly specify utilities of actions which the system tries to maximize due to the very nature of the dissipative process at optimal coupling
- Attempts under way to understand emergence of SOC (Self organized criticality) in complex systems in the current framework of dissipative open quantum systems

Thank You!